

COMPOSITION OF LORENTZ TRANSFORMATIONS IN TERMS OF THEIR GENERATORS

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Abstract

Two-forms in Minkowski space-time may be considered as generators of Lorentz transformations. Here, the covariant and general expression for the composition law (Baker-Campbell-Hausdorff formula) of two Lorentz transformations in terms of their generators is obtained. Every subalgebra of the Lorentz algebra of such generators, up to one, may be generated by a sole pair of generators. When the subalgebra is known, the above BCH formula for the two two-forms simplifies. Its simplified expressions for all such subalgebras are also given.

Key words: Lorentz transformations, Lie algebras of two-forms, Baker-Campbell-Hausdorff formula.

I Introduction

In Minkowski space-time, global Lorentz transformations are used to relate *inertial observers*. In a general space-time, local Lorentz transformations [1], along time-like congruences of curves or space-like families of hypersurfaces, are used to relate arbitrary frames to *comoving observers* [2] or to *synchronizations* [3].

In most of the problems, the transformations involved belong to the *proper orthochronous* Lorentz group (its connected component of the identity), so that they are univocally given by the exponential of the elements of the Lorentz algebra [4]. Local Lorentz transformations of the space-time may thus be given by exponentials of the two-forms [5] of the space-time.

This representation by two-forms of local Lorentz transformations has the important advantage of involving exclusively its *intrinsic elements* [6]; these are the elements in which the corresponding two-forms decompose [7]. Nevertheless, this important advantage is obscured by the practical and formal difficulties that arise in the composition of transformations, where the corresponding two-forms are related by the Baker-Campbell-Hausdorff (BCH) formula. It is the absence of a simple and compact expression for the BCH formula that originates these difficulties. The main purpose of this paper is to obtain such an expression.

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The simpler example of classical rotations illustrates clearly the above situation in the three-dimensional euclidean case. Let us remember it. The nine components of a rotation matrix R may be related in more or less complicated forms to different relative parametrizations [8], but the intrinsic elements of the matrix are the rotation axis \mathbf{u} and the rotation angle α . In the exponential domain, the element \mathbf{r} of the rotation algebra corresponding to R decomposes in the form $\mathbf{r} = \alpha * \mathbf{u}$ with $\alpha = |\mathbf{r}|$. Here $*$ is the dual operator associated to the euclidean metric δ so that, \mathbf{u} being a vector, $*\mathbf{u}$ is a two-form. And $|\mathbf{r}| = (\mathbf{r}, \mathbf{r})^{\frac{1}{2}}$ is the *module* of \mathbf{r} , with $(\mathbf{r}, \mathbf{r}) = -\frac{1}{2} \text{tr } \mathbf{r}^2$ and $\mathbf{r}\mathbf{s}$ is the induced product on two-forms (in local coordinates $(\mathbf{r}\mathbf{s})_{\mu\nu} = \mathbf{r}_{\mu\tau} \mathbf{s}_{\nu}^{\tau}$). Thus, one has for R the expression:

$$R = \exp \mathbf{r} = \delta - \frac{\sin \alpha}{\alpha} \mathbf{r} + \frac{1 - \cos \alpha}{\alpha^2} \mathbf{r}^2. \quad (1)$$

And conversely, starting from R , one obtains

$$\mathbf{r} = \log R = \frac{\arcsin \rho/2}{\rho} ({}^t R - R) \quad (2)$$

where ρ is given by

$$\rho \equiv \sqrt{(1 + \text{tr } R)(3 - \text{tr } R)}$$

and ${}^t R$ denotes the transposed of R . In other words, the rotation angle α and the rotation axis \mathbf{u} of a rotation matrix R are intrinsically given by

$$\cos \alpha = \frac{1}{2}(\text{tr } R - 1), \quad \mathbf{u} = \frac{1}{\rho} * ({}^t R - R).$$

Suppose now that we have another rotation matrix S corresponding to the rotation vector $\beta \mathbf{v}$, that is to say, to the rotation algebra element $\mathbf{s} = \beta * \mathbf{v}$. The element \mathbf{t} corresponding to the composed rotation $T = RS$ is given by the BCH-formula [9]:

$$\begin{aligned} \mathbf{t} &= \mathbf{r} \bullet \mathbf{s} \\ &= \frac{2\sigma}{\sin \sigma} \left\{ \frac{1}{\alpha} \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \mathbf{r} + \frac{1}{\beta} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \mathbf{s} + \frac{1}{\alpha\beta} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} [\mathbf{r}, \mathbf{s}] \right\}, \end{aligned} \quad (3)$$

where σ is given by

$$\cos \sigma = \cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \gamma \sin \frac{\alpha}{2} \sin \frac{\beta}{2},$$

γ being the cosine of the rotation axes, $\gamma = (\mathbf{u}, \mathbf{v}) = \frac{1}{\alpha\beta}(\mathbf{r}, \mathbf{s})$, and $[\mathbf{r}, \mathbf{s}]$ being the Lie bracket of the two-forms \mathbf{r} and \mathbf{s} , $[\mathbf{r}, \mathbf{s}] = \mathbf{r}\mathbf{s} - \mathbf{s}\mathbf{r}$.

In other words, if \mathbf{r} and \mathbf{s} are two rotations corresponding to the rotation vectors $\alpha \mathbf{u}$ and $\beta \mathbf{v}$, the rotation angle θ and the rotation axis \mathbf{w} of their composition \mathbf{t} , $\mathbf{t} = \mathbf{r} \bullet \mathbf{s} = \theta * \mathbf{w}$, are given by

$$\begin{aligned} \theta &= 2\sigma, \\ \mathbf{w} &= \frac{1}{\sin \sigma} \left\{ \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \mathbf{u} + \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \mathbf{v} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \mathbf{u} \times \mathbf{v} \right\}, \end{aligned} \quad (4)$$

where $\mathbf{u} \times \mathbf{v}$ is the vector product.

In Minkowski space-time, the analogous of expression (1), that is to say, the general and covariant explicit form of local Lorentz transformations as exponential of two-forms, has been given in [10], although some partial results were already known [11]. Nevertheless, the analogous of expression (2) for the two-forms as logarithms of local Lorentz transformations seems not to have been considered but in [10].

Here we shall obtain the analogous of expressions (3) and (4) for Minkowski space-time, that is to say, the general and covariant expression of the Baker-Campbell-Hausdorff composition \bullet of two two-forms as well as the relations between the intrinsic elements of this composition and those of its factor two-forms [12].

The BCH composition of two two-forms belongs to the algebra generated by them. The unic proper subalgebra of the three-dimensional rotation algebra being one-dimensional (rotations about same axis), one has for it $[\mathbf{r}, \mathbf{s}] = 0$, and this is the sole case in which expression (3) simplifies. The situation is drastically different for the Lorentz algebra. From its thirteen proper subalgebras, we have shown [13] that *twelve* of them may be generated by a pair of two-forms, so that the BCH formula for the Lorentz algebra may be simplified in many cases. We give here *all* these simplified expressions.

Our results are well adapted to theoretical considerations as well as to practical computations. They may be applied in all situations in which Lorentz transformations are implied, global ones in special relativity or local ones in both, special and general relativity. And this, not only for the above mentioned problems of adapted observers or synchronizations, but also in the study of special decompositions [14], Thomas precession [15], general equations of helices [16], motion of charged particles in particular electromagnetic fields [17], or the generalization of the binomial theorem [18].

These results may be also useful for heuristic researchs in other fields. For example, in non linear electromagnetic theory. Physically, algebras are seen as first (tangent) approximations or weak (little) perturbations. This suggests a guiding idea for the search of non linear electromagnetic equations: to consider that the first object to be “nonlinearized” are not Maxwell equations for the electromagnetic field, but *the electromagnetic field* itself. Being today described by a two-form (element of the Lorentz algebra), the “finite” or “strong” description of the electromagnetic field would be given by a Lorentz field tensor, its exponential [19].

The paper is organized as follows. The computation of exponentials, logarithms and BCH compositions being easier in complex spaces, Section II is devoted to remember the real and complex elements that we shall need as well as to the obtaining of the exponential of complex two-forms. Section III contains the general and covariant expression of the BCH composition of two two-forms as a linear combination of them, of their commutator and of their duals, with the coefficients depending on the invariants of the pair of two-forms. In Section IV the simplified expressions for each of the twelve proper subalgebras of the Lorentz algebra that may be generated by a pair of two-forms are obtained. In particular, half of these twelve subalgebras have the remarkable property that the eigenvalues of the BCH composition are the sum of the corresponding eigenvalues of the factor two-forms. The characterization of these subalgebras is given in this section.

II Preliminaries

We denote by $\Lambda^{\mathbb{C}}$ the complexification of Λ , space of two-forms on Minkowski space M_4 : $\Lambda^{\mathbb{C}}$ is the complex linear space associated to $\Lambda \times \Lambda$ by the complex structure $J(F, G) = (-G, F)$ for $F, G \in \Lambda$. Thus, $\Lambda^{\mathbb{C}}$ is a \mathbb{C} -linear space of complex dimension 6 with two relevant \mathbb{C} -linear subspaces

$$\begin{aligned}\Lambda^+ &= \{f \in \Lambda^{\mathbb{C}} \mid *f = if\} = \{F - i * F \mid F \in \Lambda\} \\ \Lambda^- &= \{f \in \Lambda^{\mathbb{C}} \mid *f = -if\} = \{F + i * F \mid F \in \Lambda\},\end{aligned}$$

where $*$ is the dual operator associated to the Lorentzian metric of M_4 . It is verified that $\Lambda^{\mathbb{C}} = \Lambda^+ \oplus \Lambda^-$ and $\overline{\Lambda^+} = \Lambda^-$, so

$$\dim_{\mathbb{C}} \Lambda^+ = \dim_{\mathbb{C}} \Lambda^- = 3.$$

As Λ has a structure of Lie algebra (with the commutator defined by $[F, G] = FG - GF$, where the product is defined in local coordinates by $(FG)_{\mu\nu} = F_{\mu\tau}G_{\nu}^{\tau}$), Λ^+ can be endowed with a \mathbb{C} -Lie algebra structure by the commutator in $\Lambda^{\mathbb{C}}$

$$[F + iG, H + iK] = [F, H] - [G, K] + i\{[F, K] + [G, H]\} \quad (5)$$

then, $\Lambda^{\mathbb{C}}$ is a \mathbb{C} -Lie algebra and Λ^+ is a \mathbb{C} -Lie subalgebra of $\Lambda^{\mathbb{C}}$.

Similarly, Λ^+ is a \mathbb{C} -metric linear space with the \mathbb{C} -scalar product in $\Lambda^{\mathbb{C}}$

$$(F + iG, H + iK) = (F, H) - (G, K) + i\{(F, K) + (G, H)\}, \quad (6)$$

where (\cdot, \cdot) is the induced scalar product in Λ given by $(F, G) = -(1/2) \text{tr}(FG)$, tr being the trace operator.

If $A \in \Lambda^+$, there is only one $F \in \Lambda$ such that $A = F - i * F$. When F is regular we have the decomposition $F = \alpha U - \tilde{\alpha} * U$ with $\alpha > 0$, $\tilde{\alpha} \in \mathbb{R}$ and U a unitary two-form $((U, *U) = 0$ and $(U, U) = -1)$. Then $\pm\alpha$, $\pm i\tilde{\alpha}$ are the eigenvalues of F , the pair $\{U, *U\}$ is called its *geometry*, and it is verified that $A = (\alpha - i\tilde{\alpha})(U - i * U)$. When F is null, its eigenvalues are 0, its *geometry* is $\{F, *F\}$ and it is verified that $A = F - i * F$. As, $(U - i * U, U - i * U) = -2$ and $(F - i * F, F - i * F) = 0$ for F null, every non vanishing $A \in \Lambda^+$ admits a unique decomposition of the form

$$A = \lambda_A C_A$$

where $\lambda_A \in \mathbb{C}$, its real part $\text{Re}(\lambda_A) \geq 0$ and $C_A \in \Lambda^+$ with $(C_A, C_A) \in \{-2, 0\}$, $\lambda_A = 1$ when $(C_A, C_A) = 0$.

A non vanishing $A \in \Lambda^+$ is called *regular* when $(A, A) \neq 0$ and null otherwise, C_A is its *geometry* and the complex number $a = \sqrt{-\frac{1}{2}(A, A)}$ its *invariant*. Let us note that only when A is regular the number λ_A of the above decomposition and its invariant a coincide.

Let $A, B \in \Lambda^+$ with $A = F - i * F$ and $B = G - i * G$ for $F, G \in \Lambda$. The *mixed invariants* of F and G are

$$\rho = (F, *G) \quad \text{and} \quad \sigma = (F, G).$$

The complex number k such that $-2k = (A, B)$ will be called the *mixed invariant* of A and B . It is easily verified that $k = -(\sigma - i\rho)$. For a pair of elements

A and B of Λ^+ the complex numbers a (the invariant of A), b (the invariant of B) and k (the mixed invariant of A and B) are called the *invariants of the pair* A, B .

The following expression gives the relation between the invariant of $[A, B]$ and the invariants of the pair A, B

$$([A, B], [A, B]) = 8(a^2b^2 - k^2). \quad (7)$$

This result is obtained by a straightforward computation taking into account Lemma 3 of [20].

The space of tensors with two covariant indices can be endowed with an associative algebra structure with identity element as well as a Lie algebra structure in the standard way; using linear extensions as in (5) and (6) the complexification of that space can also be endowed with an associative algebra structure with identity element and with a \mathbb{C} -Lie algebra structure. The expression for the product is

$$(M + iN)(P + iQ) = MP - NQ + i(MQ + NP).$$

From the identity $FG - *G * F = -(F, *G)g$, with F and G in Λ , it is obtained for A and B in Λ^+

$$AB - *B * A = AB + BA = -(A, B)g; \quad (8)$$

where the first equality is a consequence of the fact that $*A = iA$ for the elements of Λ^+ .

The following results are oriented to the obtaintion of the exponential of a complex two-form. Next lemma can be proven using (8) and induction over n .

Lemma 1 *For any two-form A of Λ^+ with invariant a , one has*

$$A^{2n} = a^{2n}g \quad \text{and} \quad A^{2n+1} = a^{2n}A \quad (n \in \mathbb{N}).$$

Let us define the complex functions ($z \in \mathbb{C}$)

$$C(z) = \cosh z \quad \text{and} \quad S(z) = \begin{cases} \frac{\sinh z}{z} & z \neq 0 \\ 1 & z = 0. \end{cases} \quad (9)$$

These are entire complex functions.

When $A \in \Lambda^+$ it is verified that

$$\exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{A^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{A^{2n+1}}{(2n+1)!} = \cosh A + \sinh A;$$

last equality is a consequence of the definition of the hyperbolic sine and cosine of a matrix. For these functions we have the following result.

Proposition 1 *For any two-form A of Λ^+ with invariant a , one has*

$$\cosh A = C(a)g \quad \text{and} \quad \sinh A = S(a)A.$$

It can be proven as follows:

$$\begin{aligned}\cosh A &= \sum_{n=0}^{\infty} \frac{A^{2n}}{2n!} = \left(\sum_{n=0}^{\infty} \frac{a^{2n}}{2n!} \right) g = C(a)g, \\ \sinh A &= \sum_{n=0}^{\infty} \frac{A^{2n+1}}{(2n+1)!} = \left(\sum_{n=0}^{\infty} \frac{a^{2n}}{(2n+1)!} \right) A = S(a)A.\end{aligned}$$

Then, as a corollary we obtain next theorem.

Theorem 1 *For any two-form A of Λ^+ one has*

$$\exp A = C(a)g + S(a)A,$$

where C and S are the functions (9) of the invariant a of A and g is the Lorentzian metric of the Minkowski space.

Our expression of the exponential for the Lorentz group [10] can be easily obtained from this result.

III The BCH-formula

The element $D \in \Lambda^+$ such that $\exp A \exp B = \exp D$ or, equivalently, $D = \log(\exp A \exp B)$ is given by the well known BCH-formula; this defines the so called BCH composition

$$A \bullet B = \log(\exp A \exp B).$$

Then, we may write the complex version of our main result.

Theorem 2 *The BCH composition $A \bullet B$ of two two-forms A, B in Λ^+ is given by*

$$A \bullet B = S(d)^{-1} \left\{ S(a) C(b)A + C(a) S(b)B + \frac{1}{2} S(a) S(b)[A, B] \right\}$$

where d , the invariant of the two-form $A \bullet B$, is given by

$$\cosh d = C(a) C(b) + k S(a) S(b) \quad (10)$$

a, b, k being the invariants of the pair A, B .

Proof: Let us put $D = A \bullet B$ being d its invariant then,

$$\begin{aligned}\exp D &= C(d)g + S(d)D = \exp A \exp B = (C(a)g + S(a)A)(C(b)g + S(b)B) \\ &= C(a) C(b)g + S(a) C(b)A + C(a) S(b)B + S(a) S(b)AB.\end{aligned}$$

First equality implies that we need to compute the antisymmetric part of last expression to obtain D as a function of A and B . If Z stands for the antisymmetric part, we have

$$Z = S(a) C(b)A + C(a) S(b)B + \frac{1}{2} S(a) S(b)[A, B],$$

and then $D = S(d)^{-1} \{S(a) C(b)A + C(a) S(b)B + \frac{1}{2} S(a) S(b)[A, B]\}$; so we have to obtain d as a function of the invariants of A and B . When Z is regular, we have $\sinh d C_D = z C_Z$, for C_Z and C_D , the geometries of Z and D , respectively, and z the invariant of Z . So this equation yields $C_D = C_Z$ and $\sinh d = z$. When Z is null, $z = d = 0$ and last expression is still true.

Let us obtain the invariant z as a function of a , b and k . From expressions (7) and

$$(A, [A, B]) = (B, [A, B]) = 0,$$

derived from (10) of [20], we have

$$\begin{aligned} z^2 &= -\frac{1}{2}(Z, Z) \\ &= S(a)^2 C(b)^2 a^2 + C(a)^2 S(b)^2 b^2 + 2 S(a) C(b) C(a) S(b) k - \\ &\quad S(a)^2 S(b)^2 (a^2 b^2 - k^2) \\ &= (C(a) C(b) + k S(a) S(b))^2 - 1. \end{aligned}$$

Therefore, we obtain the expression of the theorem. \square

To obtain the expression of the BCH-formula for the Lorentz group we need the following proposition whose proof is based on the fact that $[A, B] = 0$ whenever $A \in \Lambda^+$ and $B \in \Lambda^-$.

Proposition 2 *Let $F, G \in \Lambda$ and $A, B \in \Lambda^+$ be such that $A = \frac{1}{2}(F - i * F)$ and $B = \frac{1}{2}(G - i * G)$, then*

$$F \bullet G = 2 \operatorname{Re}(A \bullet B).$$

Proof: As, by definition

$$\exp F \bullet G = \exp F \exp G$$

then

$$\begin{aligned} \exp F \bullet G &= \exp(A + \overline{A}) \exp(B + \overline{B}) = \exp A \exp \overline{A} \exp B \exp \overline{B} \\ &= \exp A \exp B \exp \overline{A} \exp \overline{B} = \exp A \exp B \exp \overline{A \bullet B} \\ &= \exp A \bullet B \exp \overline{A \bullet B} = \exp A \bullet B \exp \overline{A \bullet B} \\ &= \exp \left((A \bullet B) \bullet \overline{(A \bullet B)} \right) = \exp \left((A \bullet B) + \overline{(A \bullet B)} \right); \end{aligned}$$

second equality is due to the fact that they commute, last equality is because the only terms of the BCH-series of any elements A and B with $[A, B] = 0$ is $A + B$. \square

With notation of the Theorem 2, let us define the complex functions

$$\begin{aligned} \mathcal{P} &= S(d)^{-1} S(a) C(b) \\ \mathcal{Q} &= S(d)^{-1} C(a) S(b) \\ \mathcal{R} &= \frac{1}{2} S(d)^{-1} S(a) S(b), \end{aligned} \tag{11}$$

of the invariants of the pair A, B and denote for short X_r and X_i the real and imaginary parts respectively of any complex function X . A straightforward computation gives the main result for the real case as a corollary of the previous theorem.

Theorem 3 *The BCH composition $F \bullet G$ of two real two-forms F and G on Minkowski space, is given by*

$$F \bullet G = \mathcal{P}_r F + \mathcal{Q}_r G + R_r[F, G] + \mathcal{P}_i * F + \mathcal{Q}_i * G + R_i * [F, G],$$

\mathcal{P} , \mathcal{Q} and \mathcal{R} are the functions (11) of the invariants of the pair $(1/2)(F - i * F)$, $(1/2)(G - i * G)$.

The eigenvalues α_\bullet and $\tilde{\alpha}_\bullet$ of $F \bullet G$, as well as the real and imaginary part of the functions \mathcal{P} , \mathcal{Q} and \mathcal{R} may be expressed in terms of the real invariants of the pair of two-forms F , G . In that respect let us define the auxiliary functions of the invariants of the two-forms F and G , α , $\tilde{\alpha}$, β and $\tilde{\beta}$:

$$\begin{aligned} Cc^\pm &= \cosh \frac{\alpha + \beta}{2} \cos \frac{\tilde{\alpha} + \tilde{\beta}}{2} \pm \cosh \frac{\alpha - \beta}{2} \cos \frac{\tilde{\alpha} - \tilde{\beta}}{2} \\ Ss^\pm &= \sinh \frac{\alpha + \beta}{2} \sin \frac{\tilde{\alpha} + \tilde{\beta}}{2} \pm \sinh \frac{\alpha - \beta}{2} \sin \frac{\tilde{\alpha} - \tilde{\beta}}{2} \\ Cs^\pm &= \cosh \frac{\alpha + \beta}{2} \sin \frac{\tilde{\alpha} + \tilde{\beta}}{2} \pm \cosh \frac{\alpha - \beta}{2} \sin \frac{\tilde{\alpha} - \tilde{\beta}}{2} \\ Sc^\pm &= \sinh \frac{\alpha + \beta}{2} \cos \frac{\tilde{\alpha} + \tilde{\beta}}{2} \pm \sinh \frac{\alpha - \beta}{2} \cos \frac{\tilde{\alpha} - \tilde{\beta}}{2} \end{aligned}$$

and

$$\begin{aligned} p &= -\sigma(\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) - \rho(\alpha_m \tilde{\beta}_m + \tilde{\alpha}_m \beta_m) \\ q &= -\sigma(\alpha_m \tilde{\beta}_m + \tilde{\alpha}_m \beta_m) + \rho(\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) \end{aligned}$$

where the *modular* scalars α_m y β_m are given by

$$\alpha_m \equiv \frac{\alpha}{\alpha^2 + \tilde{\alpha}^2}, \quad \beta_m \equiv \frac{\beta}{\alpha^2 + \tilde{\alpha}^2}.$$

Let us note that if f is any of the functions Cc^\pm , Ss^\pm , p or q it is verified that

$$f(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = f(\beta, \tilde{\beta}, \alpha, \tilde{\alpha}); \quad (12)$$

meanwhile, if f^\pm is any of the functions Cs^\pm or Sc^\pm , one has

$$f^\pm(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = f^\mp(\beta, \tilde{\beta}, \alpha, \tilde{\alpha}).$$

Let us introduce the functions λ , μ , l and m given by

$$\begin{aligned} \lambda &= (1/2)(Cc^+ + pCc^- + qSs^-), & \mu &= (1/2)(Ss^+ + pSs^- - qCc^-) \\ l &= \lambda^2 + \mu^2, & m &= \sqrt{(l+1)^2 - 4\lambda^2}. \end{aligned} \quad (13)$$

From Theorem 2, we have

$$\cosh \frac{1}{2}(\alpha_\bullet - i\tilde{\alpha}_\bullet) = C(a) C(b) + k S(a) S(b),$$

that is to say,

$$\lambda = \cosh \frac{\alpha_\bullet}{2} \cos \frac{\tilde{\alpha}_\bullet}{2}, \quad \mu = \sinh \frac{\alpha_\bullet}{2} \sin \frac{\tilde{\alpha}_\bullet}{2}.$$

Then, denoting by ϵ_λ and ϵ_μ respectively the signs of the above scalars λ and μ , one has

Theorem 4 *The invariants α_\bullet and $\tilde{\alpha}_\bullet$ of the BCH composition two-form $F \bullet G$ of the two two-forms F and G , are given by*

$$\cosh \alpha_\bullet = l + m, \quad \cos \tilde{\alpha}_\bullet = l - m,$$

where $\text{sign}(\sin \tilde{\alpha}_\bullet) = \epsilon_\lambda \epsilon_\mu$ and, l and m are the functions (13) of the invariants of the pair F, G .

From this theorem and property (12) one has the following result:

Corollary 1 *The invariants of the BCH compositions $F \bullet G$ and $G \bullet F$ of two two-forms F and G coincide.*

The real and imaginary part of $S(d)^{-1}$ are given, respectively, by

$$\begin{aligned} \mathcal{M} &= \frac{\alpha_\bullet \sinh \frac{\alpha_\bullet}{2} \cos \frac{\tilde{\alpha}_\bullet}{2} + \tilde{\alpha}_\bullet \cosh \frac{\alpha_\bullet}{2} \sin \frac{\tilde{\alpha}_\bullet}{2}}{\sinh^2 \frac{\alpha_\bullet}{2} \cos^2 \frac{\tilde{\alpha}_\bullet}{2} + \cosh^2 \frac{\alpha_\bullet}{2} \sin^2 \frac{\tilde{\alpha}_\bullet}{2}} \\ \mathcal{N} &= \frac{\alpha_\bullet \cosh \frac{\alpha_\bullet}{2} \sin \frac{\tilde{\alpha}_\bullet}{2} - \tilde{\alpha}_\bullet \sinh \frac{\alpha_\bullet}{2} \cos \frac{\tilde{\alpha}_\bullet}{2}}{\sinh^2 \frac{\alpha_\bullet}{2} \cos^2 \frac{\tilde{\alpha}_\bullet}{2} + \cosh^2 \frac{\alpha_\bullet}{2} \sin^2 \frac{\tilde{\alpha}_\bullet}{2}}; \end{aligned} \tag{14}$$

but from theorem 4 one has

$$\begin{aligned} \cosh\left(\frac{\alpha_\bullet}{2}\right) &= (1/\sqrt{2})\sqrt{l+m+1}, \quad \cos\left(\frac{\tilde{\alpha}_\bullet}{2}\right) = \epsilon_\lambda(1/\sqrt{2})\sqrt{l-m+1}, \\ \sinh\left(\frac{\alpha_\bullet}{2}\right) &= (1/\sqrt{2})\sqrt{l+m-1}, \quad \sin\left(\frac{\tilde{\alpha}_\bullet}{2}\right) = \epsilon_\mu(1/\sqrt{2})\sqrt{-l-m+1}, \end{aligned}$$

so that we have:

Corollary 2 *The functions \mathcal{M} and \mathcal{N} of the invariants of the pair F, G are given by*

$$\begin{aligned} \mathcal{M} &= \frac{\epsilon_\lambda}{m} \sqrt{m+n} \arg \cosh(l+m) + \frac{\epsilon_\mu}{m} \sqrt{m-n} \arg \cos(l-m) \\ \mathcal{N} &= \frac{\epsilon_\mu}{m} \sqrt{m-n} \arg \cos(l-m) + \frac{\epsilon_\lambda}{m} \sqrt{m+n} \arg \cosh(l+m). \end{aligned}$$

Then, one may obtain from (11) the following result:

Theorem 5 *The coefficients of the BCH composition $F \bullet G$ of two two-forms F and G in the expression*

$$F \bullet G = \mathcal{P}_r F + \mathcal{Q}_r G + R_r[F, G] + \mathcal{P}_i * F + \mathcal{Q}_i * G + R_i * [F, G]$$

are given by

$$\begin{aligned}
\mathcal{P}_r &= (\alpha_m \mathcal{M} - \tilde{\alpha}_m \mathcal{N}) S c^+ + (\tilde{\alpha}_m \mathcal{M} + \alpha_m \mathcal{N}) C s^+ \\
\mathcal{P}_i &= (\tilde{\alpha}_m \mathcal{M} + \alpha_m \mathcal{N}) S c^+ - (\alpha_m \mathcal{M} - \tilde{\alpha}_m \mathcal{N}) C s^+ \\
\mathcal{R}_r &= \left((\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) \mathcal{M} - (\alpha_m \tilde{\beta}_m + \tilde{\alpha}_m \beta_m) \mathcal{N} \right) C c^- + \\
&\quad \left((\alpha_m \tilde{\beta}_m + \tilde{\alpha}_m \beta_m) \mathcal{M} + (\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) \mathcal{N} \right) S s^- \\
\mathcal{R}_i &= \left((\alpha_m \tilde{\beta}_m + \alpha_m \tilde{\beta}_m) \mathcal{M} + (\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) \mathcal{N} \right) C c^- - \\
&\quad \left((\alpha_m \beta_m - \tilde{\alpha}_m \tilde{\beta}_m) \mathcal{M} - (\alpha_m \tilde{\beta}_m + \tilde{\alpha}_m \beta_m) \mathcal{N} \right) S s^- \\
\mathcal{Q}_r(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) &= \mathcal{P}_r(\beta, \tilde{\beta}, \alpha, \tilde{\alpha}) \\
\mathcal{Q}_i(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) &= \mathcal{P}_i(\beta, \tilde{\beta}, \alpha, \tilde{\alpha}).
\end{aligned}$$

One could directly obtain the geometry of $F \bullet G$ as a function of the invariants and the geometries of the pair F, G . In the singular case, the two-form and its geometry are proportional, so that we have only to study the regular case. Both cases are discriminated by the value of $C(a) C(b) + k S(a) S(b)$; from Theorem 2, this value is 1 only when $F \bullet G$ is singular or 0. Taking into account Proposition 2, that is to say,

$$A \bullet B = \frac{1}{2} \{ F \bullet G - i * (F \bullet G) \},$$

and the definition of the geometry $\{W_\bullet, *W_\bullet\}$ of $F \bullet G$,

$$F \bullet G = \alpha_\bullet W_\bullet - \tilde{\alpha}_\bullet * W_\bullet$$

one has

$$A \bullet B = \frac{1}{2} (\alpha_\bullet - i \tilde{\alpha}_\bullet) (W_\bullet - i * W_\bullet);$$

then, from theorem 2 we obtain

Theorem 6 *The element W_\bullet of the geometry $\{W_\bullet, *W_\bullet\}$ of the BCH composition $F \bullet G$ of two two-forms F and G , is given by the same expression of Theorem 5 for $F \bullet G$ where the functions (14) \mathcal{M} and \mathcal{N} are substituted respectively by*

$$\frac{1}{2} (\mathcal{M} - \mathcal{N}) \quad \text{and} \quad \frac{1}{2} (\mathcal{M} + \mathcal{N}).$$

IV Reduction of the BCH-formula for each type of subalgebra

We have shown elsewhere [13] that from the thirteen proper subalgebras of the Lorentz algebra, twelve of them may be generated by a pair of two-forms, so that the general expression of the BCH composition given above may be simplified in these twelve cases. It is the purpose of this Section to obtain the corresponding simplified expressions.

Half of these twelve cases have in common a particular property: *the eigenvalues $\alpha_\bullet, \tilde{\alpha}_\bullet$ of the BCH composition $F \bullet G$ are the sum of the corresponding eigenvalues $\alpha, \tilde{\alpha}$ of F and $\beta, \tilde{\beta}$ of G :*

$$\alpha_\bullet = \alpha + \beta, \quad \tilde{\alpha}_\bullet = \tilde{\alpha} + \tilde{\beta}. \quad (15)$$

We shall begin by obtaining some common properties to these cases. From (15), expressions (11) simplify to:

$$\begin{aligned} \mathcal{P} &= \frac{(a+b) \sinh a \cosh b}{a \sinh(a+b)} \\ \mathcal{Q} &= \frac{(a+b) \cosh a \sinh b}{b \sinh(a+b)} \\ \mathcal{R} &= \frac{1}{2} \frac{(a+b) \sinh a \sinh b}{ab \sinh(a+b)}. \end{aligned} \quad (16)$$

So, we do not have to use (10) to obtain these coefficients. This expression implies that $d = a + b$ iff $k/ab = 1$. But

$$k = -\lambda_A \lambda_B (\sigma' - \rho'),$$

where λ_A is the invariant of A when A is regular and 1 otherwise, $\sigma' = (U, V)$ and $\rho' = (U, *V)$ when $A = \frac{1}{2}(F - i * F)$ and $B = \frac{1}{2}(G - i * G)$. Therefore,

$$\cosh d = \begin{cases} \cosh a \cosh b - (\sigma' - i\rho') \sinh a \sinh b & \text{when both are regular} \\ \cosh a - (\sigma' - i\rho') \sinh a & \text{when only } A \text{ is regular} \\ -(\sigma' - i\rho') & \text{when both are null.} \end{cases}$$

Hence, $d = a + b$ iff $\rho' = 0$ and $\sigma' = -1$ with $ab \neq 0$, $\sigma' = 0$ with $ab = 0$. These situations may be related to the relative positions of the non space like planes asociated to the geometries of the two-forms, that is, the pair $(\pi(U), \pi(V))$. Relative positions of pairs of planes were studied in [13]. With that notation, the above situations correspond to the following relative positions:

$$\Pi_{1,1} \Pi_{1,2} \Pi_{2,1}, \Pi_{2,2}, \Pi_{2,2}, \Pi_{2,3}.$$

But these are the sole relatives positions of a pair of planes having at least one common null direction. Following the Schell classification of subalgebras of the Lorentz group [21] (see Table 1 for the relation of this notation and that of Patera *et al.* [22]), our Theorem 3 of [13] implies that a pair of two-forms, whose non spacelike planes asociated with their geometries have one of those positions, generates one and only one of the following types of subalgebra of dimension greater than two:

$$R_6, R_7, R_8, R_{11}, R_{12}, R_{14}.$$

On the other hand, it may be verified that these are the sole algebras generated by two two-forms whose Killing–Cartan form is singular. Thus, denoting by $A(F, G)$ the algebra generated by F and G , one has

Theorem 7 *Let $\alpha, \tilde{\alpha}$ be the eigenvalues of F , $\beta, \tilde{\beta}$ those of G , and $\alpha_\bullet, \tilde{\alpha}_\bullet$ those of their BCH composition $F \bullet G$, and let $\{U, *U\}$ and $\{V, *V\}$ be the geometries of F and G , respectively. Then, the following conditions are equivalent:*

Dimension	Type		Basis
0	R_1	F_{15}	
1	R_2	F_{13}	A_1
1	R_3	F_{14}	A_3
1	R_4	F_{12}	A_2
1	R_5	F_{11}	$A_1 + \tau A_2$ ($\tau \neq 0$)
2	R_6	F_8	A_1, A_3
2	R_7	F_9	A_1, A_2
2	R_8	F_{10}	A_3, A_4
3	R_9	F_7	A_1, A_3, A_4
3	R_{10}	F_4	A_1, A_3, A_5
3	R_{11}	F_6	A_2, A_3, A_4
3	R_{12}	F_5	$A_1 + \tau A_2, A_3, A_4$ ($\tau \neq 0$)
3	R_{13}	F_3	$A_2, A_3 - A_5, A_4 - A_6$
5	R_{14}	F_2	A_1, A_2, A_3, A_4
6	R_{15}	F_1	$A_1, A_2, A_3, A_4, A_5, A_6$

Table 1: Subalgebras of the Lie algebra of Lorentz group

1. $\alpha_\bullet = \alpha + \beta$ and $\tilde{\alpha}_\bullet = \tilde{\alpha} + \tilde{\beta}$,
2. planes $\pi(U)$ and $\pi(V)$ have at least one common null direction,
3. the relative position of the pair of plane $(\pi(U), \pi(V))$ is one and only one of the following: $\Pi_{1,1}$ $\Pi_{1,2}$ $\Pi_{2,1}$ $\Pi_{2,2}$ $\Pi_{2,2}$ $\Pi_{2,3}$,
4. the algebra $\mathcal{A}(F, G)$ is one of the following ones:
 $R_6, R_7, R_8, R_{11}, R_{12}, R_{14}$,
5. the Killing–Cartan form of $\mathcal{A}(F, G)$ is singular.

The subalgebras of the Lie algebra of Lorentz group are given in Table 1. Each type corresponds to a class of conjugation by the orthochronous proper Lorentz group (connected component of the identity), except for the types where it appears the real number τ . In these cases each τ defines a different conjugation class. The R 's and the F 's columns of Table 1 correspond, respectively, to notations by Schell (see [21]) and by Patera et al. (see [22]). The last column includes a basis of each type in terms of the two-forms associated to a real null tetrad $\{l, m, p, q\}$ (that is to say a tetrad verifying $(l, l) = (m, m) = (l, p) = (l, q) = (m, p) = (m, q) = (p, q) = 0$ and $(l, m) = -(p, p) = -(q, q) = 1$); we have made use of the notation:

$$\begin{aligned} A_1 &= l \wedge m & A_3 &= l \wedge p & A_5 &= m \wedge p \\ A_2 &= p \wedge q & A_4 &= l \wedge q & A_6 &= m \wedge q. \end{aligned}$$

From now on it is considered that

$$A = \frac{1}{2}(F - i * F) \quad \text{and} \quad B = \frac{1}{2}(G - i * G).$$

When $\mathcal{A}(F, G)$ is a R_7 or R_8 algebra then $[F, G] = 0$ and $[A, B] = 0$. In these cases $d = a + b$ (Theorem 7). For R_8 both are null and then $\mathcal{P} = \mathcal{Q} = \mathcal{R} = 1$; therefore, $F \bullet G = F + G$. For R_7 both are regular and the geometries of A and

B can be taken such that $C_A = C_B$ (Proposition 8 of [20]); hence $bA = aB$. Therefore,

$$A \bullet B = \frac{a+b}{\sinh(a+b)} \left\{ \frac{\sinh a}{a} \cosh b A + \cosh a \frac{\sinh b}{b} B \right\} = \left(1 + \frac{b}{a} \right) A = A + B.$$

Thus, the well-known result for commuting two-forms is obtained;

Theorem 8 *When $\mathcal{A}(F, G)$ is a R_7 or R_8 algebra one has $F \bullet G = F + G$.*

Suppose $\mathcal{A}(F, G)$ is a R_6 algebra then $\dim \mathcal{A}(F, G) = 2$ and $[F, G] \neq 0$. Thus, from Theorem 3 of [13], F and G are simple two-forms, that have only one common principal direction and at least one non common principal one. So, at least one is regular (say F), a , b and k are real and, by Theorem 7, $d = a + b$. If α is the non null eigenvalue of F a straightforward computation yields

$$[F, G] = \frac{\sigma}{\alpha} F + \alpha G. \quad (17)$$

Let us remark, that $\sigma = -1$ when both two-forms are regular and $\sigma = 0$ when one is null (G in this case). Taking into account expressions (16) and (17) the following theorem is obtained,

Theorem 9 *When $\mathcal{A}(F, G)$ is a R_6 algebra and F is the regular two-form one has*

$$F \bullet G = \frac{\alpha + \beta}{\sinh \frac{\alpha + \beta}{2}} \left\{ \frac{\sinh \frac{\alpha}{2}}{\alpha} \left(\cosh \frac{\beta}{2} - \sinh \frac{\beta}{2} \right) + \frac{\sinh \frac{\beta}{2}}{\beta} \left(\cosh \frac{\alpha}{2} + \sinh \frac{\alpha}{2} \right) G \right\}$$

where α is the non null eigenvalue of F and β the eigenvalue of G .

For R_{10} , R_{11} and R_{13} algebras we know from Theorem 3 of [13] that F and G are simple and $(F, *G) = 0$. First condition implies that a and b are pure real or imaginary numbers; then functions $C(a)$, $S(a)$, $C(b)$ and $S(b)$ are real valued. Second condition implies that k is real. Moreover, $d/\sinh d$ is real. Let us show this. As $F \bullet G$ is a linear combination of F , G and $[F, G]$, $(F \bullet G, *F \bullet G) = 0$ because $[F, G]$ is simple (Proposition 9 of [20]) and $[F, [F, G]] = [F, *[F, G]] = 0$; thus d is a pure real or imaginary number which implies that $d/\sinh d$ is real. Therefore, \mathcal{P} , \mathcal{Q} and \mathcal{R} are real valued functions of complex variables.

Theorem 10 *When $\mathcal{A}(F, G)$ is a R_{10} , R_{11} or R_{13} algebra, one has*

$$F \bullet G = \mathcal{P}F + \mathcal{Q}G + \mathcal{R}[F, G].$$

Let us remark that this expression is equivalent to (3) for R_{13} , this algebra being isomorphic to the Lie algebra of the rotation group in \mathbb{R}^3 .

Consider $\mathcal{A}(F, G)$ is a \mathcal{R}_{12} algebra. In this case (Theorem 3 of [13]) F and G are not both simple, they have one common principal direction and, either their eigenvalues are proportional or one is null and the other is not simple. Assume F is not simple. Therefore, $d = a + b$ (Theorem 7) and there is a real number η such that $b = \eta a$ ($\eta = 0$ when G is null). Taking into account (16), one has

$$\begin{aligned} \mathcal{P} &= \frac{(1 + \eta) \sinh a \cosh \eta a}{\sinh(1 + \eta)a} \\ \mathcal{Q} &= \frac{(1 + \eta) \cosh a \sinh \eta a}{\eta \sinh(1 + \eta)a} \\ \mathcal{R} &= \frac{1}{2} \frac{(1 + \eta) \sinh a \sinh \eta a}{\eta a \sinh(1 + \eta)a}. \end{aligned}$$

Thus $\mathcal{P} + \eta\mathcal{Q} = 1 + \eta$ which implies

$$\mathcal{P}_r = 1 + \eta - \eta\mathcal{Q}_r, \quad \mathcal{P}_i = -\eta\mathcal{Q}_i. \quad (18)$$

For the non spacelike part of the geometries of F and G , U and V one obtain, from (17), that $[U, V] = \sigma'U + V$ where $\sigma' = (U, V)$. As $b = \eta a$, the commutator of F and G verifies

$$[F, G] = -\alpha(\eta F - G) + \tilde{\alpha} * (\eta F - G). \quad (19)$$

Combining (18) and (19) we obtain the following result.

Theorem 11 *Let $\mathcal{A}(F, G)$ be a \mathcal{R}_{12} algebra, and η the real number such that $\beta - i\tilde{\beta} = \eta(\alpha - i\tilde{\alpha})$ for the eigenvalues of G and F ; when F is not simple one has*

$$\begin{aligned} F \bullet G &= \left\{ 1 - \eta \left(\mathcal{Q}_r + \frac{\alpha}{\tilde{\alpha}} \mathcal{Q}_i + \frac{\alpha^2 + \tilde{\alpha}^2}{\tilde{\alpha}} \mathcal{R}_i - 1 \right) \right\} F + \\ &\quad \left\{ \mathcal{Q}_r + \frac{\alpha}{\tilde{\alpha}} \mathcal{Q}_i + \frac{\alpha^2 + \tilde{\alpha}^2}{\tilde{\alpha}} \mathcal{R}_i \right\} G - \left\{ \frac{1}{\tilde{\alpha}} \mathcal{Q}_i - \mathcal{R}_r + \frac{\alpha}{\tilde{\alpha}} \mathcal{R}_i \right\} [F, G]. \end{aligned}$$

When $\mathcal{A}(F, G)$ is a R_{14} algebra, expression (17) still holds for the non space-like part of the geometries of F and G ; now it is verified that $\sigma_{UV} = -1$; thus

$$[F, G] = -\beta F + \alpha G + \tilde{\beta} * F - \tilde{\alpha} * G.$$

Therefore, we have the following result.

Theorem 12 *Let $F = \alpha U - \tilde{\alpha} * U$ and $G = \beta V - \tilde{\beta} * V$ be such that $\mathcal{A}(F, G)$ is a R_{14} algebra then one has*

$$\begin{aligned} F \bullet G &= (\mathcal{P}_r - \beta\mathcal{R}_r - \tilde{\beta}\mathcal{R}_i)F + (\mathcal{Q}_r + \alpha\mathcal{R}_r + \tilde{\alpha}\mathcal{R}_i)G + \\ &\quad (\mathcal{P}_i + \tilde{\beta}\mathcal{R}_r - \beta\mathcal{R}_i) * F + (\mathcal{Q}_i - \tilde{\alpha}\mathcal{R}_r + \alpha\mathcal{R}_i) * G. \end{aligned}$$

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- [1] Here a *local Lorentz transformation* means a field of Lorentz transformations of the tangent space at *each* point.
- [2] That is to say, to observers whose velocity vector is tangent to the curves of the congruence.
- [3] A *synchronization* is a specification of the locus of points (hypersurfaces) of equal time. Every synchronization has a *natural* set of observers: those whose velocity vector is normal to the family of hypersurfaces.

- [4] They are given biunivocally only for *exponential* groups (J. Dixmier, Bull. Soc. math. France, **85**, 113 (1957); L. Pukanszky, Trans. Amer. Math. Soc., **126**, 487 (1967).
- [5] Because of the space-time metric, the elements of the algebra may be written as second order antisymmetric covariant tensors at each point so that, in the corresponding domain of the space-time, they define a two-form.
- [6] The usual representation of Lorentz transformations by matrices or second order tensors carries an excessive number of nonstrict quantities, namely $n^2 = 16$ components, which hides their *intrinsic elements*. These elements depend only on $n(n - 1)/2 = 6$ parameters, the group dimension, and are those in which the Lorentz transformations may be *biunivocally* and *covariantly* decomposed. The intrinsic elements of a Lorentz transformation are thus its non space-like invariant 2-plane, and its two eigenvalues. As it is well known, in the regular case one of these eigenvalues is a hyperbolic angle, and gives the magnitude of the *proper* boost on the timelike invariant 2-plane. The other eigenvalue is a trigonometric angle, and fixes the rotation on the orthogonal space-like invariant 2-plane. Intrinsic elements have not to be confused with velocity-rotation relative parametrizations, for which biunivocality fails.
- [7] See next Section.
- [8] Euler angles, Cayley-Klein parameters, etc.
- [9] For the original articles see: J.E. Campbell, Proc. London Math. Soc., **28**, 381 (1897); **29**, 14 (1898); H. F. Baker, *ibid*, **34**, 347 (1902); **2**, 293 (1904); **3**, 24 (1905); F. Hausdorff, Ber. Verhandl. Saechs. Akad. Wiss. Leipzig, Math. Naturw. Kl., **58**, 19 (1906).
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- [11] A. H. Taub, Phys. Rev., **73**, 786 (1948); S. L. Bazanski, J. Math. Phys., **6**, 1201 (1965);
- [12] It is to be noted that the product of two Lorentz transformations in terms of *relative* parameters (namely, relative velocity and relative rotation), which is well known from long time (see, for exemple M. Rivas *et al.*, Eur. J. Phys., **7**, 1 (1986)), differs strongly from the BCH product. This is due to the facts that in the case of relative parametrizations every factor is framed in a different basis (i.e. for different observers) and that the corresponding parametrizations refers to these different bases. A connection between the relative product and the BCH formula not only needs the relation between relative parameters and intrinsic elements, but *also* the relation between the relative parameters with respect to different observers, involving notions such as “velocity of a point with respect to an observer *as seen* by another observer”. We shall not consider here such a connection.
- [13] B. Coll and F. San José, *On the algebras generated by two 2-forms in Minkowski space-time*, J. Math. Phys., **37**, 5792 (1996); see also *Relative position of a pair of planes and algebras generated by two 2-forms in relativity*, in *Recent Developments in Gravitation*, World Scientific, 1991, p. 210.

- [14] See for example C. B. van Wyk, J. Math. Phys., **32**, 425 (1991).
- [15] Corrections to the Tomas precession (L. H. Thomas, *Nature*, **117**, 514 (1926)) have been obtained from the BCH formula by N. Salingaros, J. Math. Phys., **25**, 706 (1984).
- [16] The equations of relativistic helices are due to J. L. Synge, Proc. Roy. Irish Acad., sec. A, **65**, 27 (1967); their expressions may be manifestly simplified using the results presented here.
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- [18] See J. Morales and A. Flores-Riveros, J. Math. Phys., **30**, 393 (1989).
- [19] The analysis of this idea needs, beside our results, their reciprocals: the internal operations on the group of Lorentz tensors that correspond by the exponential to the two internal operations of its algebra (addition and commutator). But this is another affair.
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